

Wigner-PDC description of photon entanglement as a local-realistic theory

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Regardless of whether the zeropoint field (ZPF) of electromagnetic vacuum fluctuations is considered as real or as merely virtual, here we show that the Wigner description of PDC-generated (Parametric Down Conversion) photon-entanglement (developed in [3–9]) can be formulated as entirely local-realistic. This involves reinterpreting the expressions for the detection probabilities, by means of an additional mathematical manipulation; though such manipulation seemingly provides enough freedom to guarantee consistency with the expectable, experimentally testable behavior of detectors, this is, in any case, irrelevant in relation to our main result, of a purely mathematical nature. This said, of course the PDC model is by construction restricted to a certain subset of QED-states, obtained directly from a mix of the vacuum state (hence one with positive Wigner function) with a quasi-classical signal (the laser), and a time-evolution governed by a quadratic Hamiltonian (hence one that preserves the positivity of the Wigner function): it is only for that subset that the local-realistic analogy can be implemented. Some additional questions are also addressed; amongst them, we provide an extremely simple interpretation on that apparently awkward “subtraction” of the average ZPF intensity at the Wigner image of the detection process, an element stemming from Glauber’s purely quantum mechanical expression. On the other hand, the immediate connection of this model with the phenomenon of “detection probability enhancement”, with the potential to reconcile local realism with the experimental violations of inhomogeneous Bell inequalities, cannot not be considered anything short of strong evidence of its physical relevance.

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I. INTRODUCTION

The truth is that so far no conclusive proof of the violation of local-realism has ever been obtained (in this regard, the detection loophole is also present in the latest important development [1]); perhaps it is time to consider the possibility that not all states allowed by QM may have a physical counterpart, in particular those that would yield correlations defying local-realism. Perhaps is also time to start giving credit to models that, even though being quantum-mechanical, may include that restriction built within their formulation; here we propose a step in this direction. This step is necessary, but of course not conclusive: in coherence with what we just said, what we treat here is just that, a “model”. We hope, nevertheless, that it serves the purpose of encouraging further work on the issue.

The Wigner picture of Quantum Optics of photon-entanglement generated by Parametric Down Conversion [2] was developed some years ago in a series of papers [3–9]; recently, the approach has been revitalized producing another stream of very interesting results [10, 11]. It departs from an stochastic electrodynamical description (hence, based on continuous variables: electromagnetic fields defined at each and every point of space), and by use of the so-called Wigner transformation [12] it acquires a form where all expectation values depend on a certain probability distribution for the value of those

fields in the vacuum. Such a picture clearly differs, on the other hand, from the usual one in Quantum Information (QInf), based on a discrete description of the photon (a particle) and not on a set of fields in a continuum space. For instance, in a polarizing beam splitter (PBS), empty polarization channels at either of the exits of the device are filled with the random components of the vacuum Zero-Point field (ZPF); these random components can for instance give rise to the enhancement of the detection probability for certain realizations of the state of the fields (certain photons, in the QED language): see note [13]. Phenomena such as this last come as a big difference with the customary model of the set polarizer-detector is treated just as a “black box”, able to extract polarization information in principle without the (explicit) intervention of any additional noise.

The Wigner-PDC framework provides an alternative, apparently local-realistic explanation of the results of many typically quantum experiments, the result of a measurement depending on (and only on) the set of hidden variables (HV) inside its light cone: in this case, both the signal propagating from the source and the additional noise introduced by the ZPF at intermediate devices and detectors. Within the first series of papers [3–9] those experiments included: frustrated two photon creation via interference [3]; induced coherence and indistinguishability in two-photon interference [3, 4]; Rarity and Tapster’s 1990 experiment with phase-momentum entanglement [4]; Franson’s (original, 1989) experiment [4]; quantum *dispersion cancellation* and Kwiat, Steinberg and Chiao’s *quantum eraser* [5]. From the most recent one [10, 11] we can also add, on one side, amongst quantum cryptography experiments based on PDC: two-

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qubit entanglement and cryptography [10] and quantum key distribution and eavesdropping [10]; on the other, an interpretation for the experimental partial measurement of the Bell states generated from a single degree of freedom (polarization) in [11]. We have also had access to some promising (but still unpublished) work on *hyper-entanglement* [14].

Perhaps needless to say, this "local realistic" picture of the quantum experiments takes place within the limits (detection efficiencies) where it is already well acknowledged that such an explanation may exist; moreover, those limits arise as natural consequences of the theory: they are simply limiting values of detection rates for states of light that are ultimately generated from the vacuum. This last feature is what makes the Wigner approach so interesting; on the other hand, above we have used "apparently" because that local-realistic interpretation, even within the corresponding limits of detection rates, was until now not devoid of difficulties, in particular related to the detection model (so far merely a one-to-one counterpart with Glauber's original expressions [15]): as a result of the normal order of operators, there average intensity due to vacuum fluctuations is "subtracted", a subtraction that seems to introduce problems related to the appearance of what could be interpreted as "negative probabilities". This is the issue that interests us here, one that, as expectable, motivated the proposal of several modifications upon the expressions for the detection probabilities: for instance, as early as in [6], "our theory is also in almost perfect one-to-one correspondence with the standard Hilbert-space theory, the only difference being the modification in the detection probability that we proposed in relation...". Those modifications ranged from the mere inclusion of temporal and spatial integration [6] to the proposal of much more complicated functional dependencies [9, 16, 17], all of them seeming to pose their own problems; for instance in [16], a departure from the quantum predictions at low or high intensities.

Our route here is a different one however: we do not propose any modification of the initial expressions for the detection probabilities (in one-to-one correspondence with the initial quantum electrodynamical model), but just explore the possibility of performing some convenient mathematical manipulation that casts them in a form consistent with the axioms of probability, hence one consistent with local-realism. Following for instance [6] (see also [15]), quantum mechanical detection probabilities, single and joint, can respectively be expressed as

$$P_i \propto \langle I_i - I_{0,i} \rangle \\ = \int_{\alpha, \alpha^*} (I_i(\alpha, \alpha^*) - I_{0,i}) W(\alpha, \alpha^*) d\alpha d\alpha^*, \quad (1)$$

$$P_{i,j} \propto \int_{\alpha, \alpha^*} (I_i(\alpha, \alpha^*) - I_{0,i}) \cdot (I_j(\alpha, \alpha^*) - I_{0,j}) \\ \times W(\alpha, \alpha^*) d\alpha d\alpha^*, \quad (2)$$

where α, α^* are vacuum amplitudes of the (relevant set

of) frequency modes [18] at the entrance of the crystal, $W(\alpha, \alpha^*)$ is obtained as the Wigner transform of the vacuum state, $I_i(\alpha, \alpha^*)$ is the field intensity (for that mode) and $I_{0,i}$ the mean intensity due to the vacuum amplitudes, both at the entrance of the i -th detector (see [18]):

$$I_0 = \int_{\alpha, \alpha^*} I_0(\alpha, \alpha^*) W(\alpha, \alpha^*) d\alpha d\alpha^*. \quad (3)$$

The last three expressions would in principle allow us to identify the vacuum amplitudes with a vector of hidden variables $\lambda \in \Lambda$ (Λ is the space of events or probabilistic space), with an associated density function $\rho(\lambda)$,

$$\lambda \equiv \alpha, \alpha^*, \quad \rho(\lambda) \equiv W(\alpha, \alpha^*). \quad (4)$$

Now, for instance in [6] it is already acknowledged that (1)–(2) cannot, because of the possible negativity of the difference $I_i(\alpha, \alpha^*) - I_{0,i}$, be written as

$$P_i = \int_{\Lambda} P_i(\det|\lambda) \rho(\lambda) d\lambda, \quad (5)$$

$$P_{i,j} = \int_{\Lambda} P_{i,j}(\det|\lambda) \rho(\lambda) d\lambda, \quad (6)$$

where naturally $P_i(\det|\lambda), P_{i,j}(\det|\lambda)$ should stay positive (or zero) always. This last is our point of departure: in this paper we propose a reinterpretation of the former marginal and joint detection probabilities based on a certain manipulation of expressions (1)–(2).

The paper is organized as follows. In Sec. II we will consider a setup with just a source and two detectors; once that is understood, the interposition of other devices between the source and the detectors poses no additional conceptual difficulty though it is nevertheless convenient to address it in some detail: this will be done in Sec. III. The calculations in these two sections find support on the proofs provided in Appendix 1, and stand for our main result in this paper. Sec. IV explores the question of "α-factorability" in the model, not only from the mathematical point of view but also providing some more physical insights on its implications.

Up to that point the novel points of the paper are made and its results are self-contained; it is nevertheless natural to extend our analysis to some of their further implications (amongst these, the consequences for Bell tests of local-realism), in Sec. V, where we also include a preliminary approach to questions regarding the physics of the real detectors. Finally, overall conclusions are presented in Sec. VI, and some supplementary material is provided in Appendix 2, which may not only help make the paper self-contained but perhaps also contribute to clarify some of the questions addressed, in particular the non-factorability issue.

II. REINTERPRETING DETECTION PROBABILITIES

Our reinterpretation of expressions (1)–(2) involves two steps:

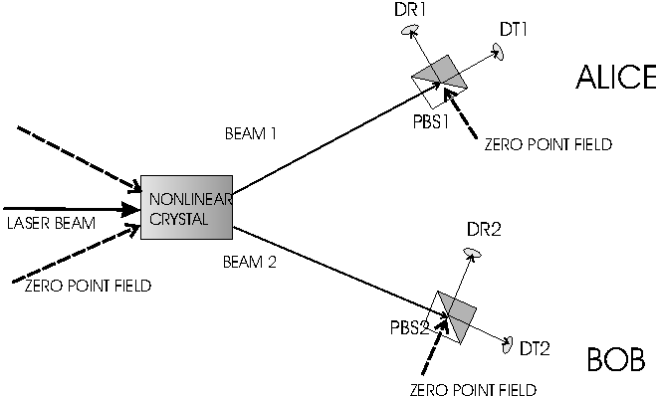


FIG. 1: Wigner-PDC scheme: photon pair generation, polarizing beam splitters (PBS) and detectors. Only relevant inputs of Zero-Point vacuum field (ZPF) are represented in the picture: “relevant” can be understood, in a classical wave-like approach, as “necessary to satisfy energy-momentum conservation” for the (set of) frequency modes of interest; in a purely quantum electrodynamical one we would be talking about conservation of the commutation relations at the empty exit channels of the devices. Besides, those new ZPF components introduced at the empty exit channels of PBS’s 1 and 2 can alter the detection probability for the signal arriving to the detectors (for instance giving rise to its “enhancement”); that signal is on the other hand determined (amongst other hidden variables) by the ZPF components entering the crystal. Figure: courtesy of A. Casado.

(A) Knowing that the following equality holds (from here on we drop detector indexes when unnecessary), for some real constant $K_{(m)}$ (“marginal”),

$$K_{(m)} \int_{\alpha, \alpha^*} (I(\alpha, \alpha^*) - I_0) W(\alpha, \alpha^*) d\alpha d\alpha^* = \int_{\Lambda} P(det|\lambda) \rho(\lambda) d\lambda, \quad (7)$$

we realize we do not need to assume

$$P(det|\lambda) \equiv K_{(m)} \cdot (I(\alpha, \alpha^*) - I_0), \quad (8)$$

as a necessary, compulsory choice; it would be enough to find some $f(x) \geq 0$, satisfying

$$K_{(m)} \int_{\alpha, \alpha^*} (I(\alpha, \alpha^*) - I_0) W(\alpha, \alpha^*) d\alpha d\alpha^* = \int_{\alpha, \alpha^*} f(I(\alpha, \alpha^*)) W(\alpha, \alpha^*) d\alpha d\alpha^*, \quad (9)$$

so we can then safely identify

$$P(det|\lambda) \equiv f(I(\alpha, \alpha^*)), \quad (10)$$

with $f(I(\alpha, \alpha^*)) \geq 0, \forall I(\alpha, \alpha^*)$. This last is nothing but solving a linear system with only one restriction and an infinite number of free parameters, whose subspace

of solutions intersects the region $f(I(\alpha, \alpha^*)) \geq 0 \forall \alpha, \alpha^*$: see Appendix 1.

Of course, once at this point, joint detection probabilities would in principle impose additional restrictions on $f(x)$; for detectors i, j we would have, for some constant $K_{(j)}$ (“joint”),

$$\begin{aligned} K_{(j)} \int (I_i(\alpha, \alpha^*) - I_{0,i}) \cdot (I_j(\alpha, \alpha^*) - I_{0,j}) \\ \times W(\alpha, \alpha^*) d\alpha d\alpha^* \\ = \int f(I_i(\alpha, \alpha^*)) \cdot f(I_j(\alpha, \alpha^*)) W(\alpha, \alpha^*) d\alpha d\alpha^*. \end{aligned} \quad (11)$$

(B) In principle, it is unclear whether all possible $f(x)$ ’s satisfying (9) can be made to meet (11), therefore we propose: instead of demanding (11) from the beginning, we define a new function $\Gamma(x, y)$, so that

$$\begin{aligned} K_{(j)} \int (I_i(\alpha, \alpha^*) - I_{0,i}) \cdot (I_j(\alpha, \alpha^*) - I_{0,j}) \\ \times W(\alpha, \alpha^*) d\alpha d\alpha^* \\ = \int \Gamma(I_i(\alpha, \alpha^*), I_j(\alpha, \alpha^*)) W(\alpha, \alpha^*) d\alpha d\alpha^*, \end{aligned} \quad (12)$$

so we can now identify

$$P_{i,j}(det|\lambda) \equiv \Gamma(I_i(\alpha, \alpha^*), I_j(\alpha, \alpha^*)). \quad (13)$$

In Appendix 1 we have shown that it is always possible to find some suitable $0 \leq \Gamma(x, y) \leq 1$, for all $x = I_i(\alpha, \alpha^*)$ and $y = I_j(\alpha, \alpha^*)$, i.e., for all possible pairs α, α^* .

The absence of factorability on the α ’s, i.e., the fact that (we may have)

$$\Gamma(I_i(\alpha, \alpha^*), I_j(\alpha, \alpha^*)) \neq f(I_i(\alpha, \alpha^*)) \cdot f(I_j(\alpha, \alpha^*)), \quad (14)$$

may come as a surprise to some, given than Clauser-Horne factorability [19] on the hidden variable λ is usually taken for granted; this is a mistake [20]. We have tried to clarify the whole issue in Appendix 2: in general, we have shown that that non-factorability, aside from perfectly legitimate from the mathematical perspective, is in this case possibly related with the need to include new hidden variables in the model. ■

It is now convenient, for the following sections, to make our notation lighter, with $\alpha \equiv \alpha, \alpha^*$, and also to define

$$\hat{f}_i(\alpha) \equiv f(I_i(\alpha)), \quad (15)$$

$$\hat{\Gamma}_{i,j}(\alpha) \equiv \Gamma(I_i(\alpha), I_j(\alpha)). \quad (16)$$

III. ADDING INTERMEDIATE DEVICES: POLARIZERS, PBS’S...

Once we place one or more devices between the crystal and the detectors, typically polarizers, polarizing beam

splitters (PBS) or other devices to allow polarization measurements (such as in Fig. 1), in general we cannot any longer describe the fields between both with only one set $\{\alpha\}$ of mode-amplitudes; we need to redefine our α 's as now associated to a particular position \mathbf{r} (they do not any longer determine a frequency mode for all space [18]). Hence, we will now have

$$\alpha(\mathbf{r}) \equiv \{\alpha_{\mathbf{k},\gamma}(\mathbf{r}), \alpha_{\mathbf{k},\gamma}^*(\mathbf{r})\}, \quad (17)$$

and, letting \mathbf{r}_s be the position of the source (the crystal), and \mathbf{r}_i the position of the i -th polarizer or PBS (or any other intermediate device), we will also redefine

$$\alpha_s \equiv \alpha(\mathbf{r}_s), \quad \alpha_i \equiv \alpha(\mathbf{r}_i), \quad (18)$$

with α_i including the *relevant* amplitudes at the (empty) exit channels of that i -th intermediate device. With α_s, α_i corresponding each to (a set of) modes with different (sets of relevant) wavevectors $\{\mathbf{k}_s\}$ and $\{\mathbf{k}_i\}$ [21], we can then regard them as two (sets of) statistically independent random variables, with all generality. The intensity at the entrance of detector i -th will therefore depend now not only on α_s but also on α_i :

$$\begin{aligned} P_i &\propto \langle I_i(\alpha_s, \alpha_i) - I_{0,i} \rangle \\ &= \int_{\alpha_s} \int_{\alpha_i} (I_i(\alpha_s, \alpha_i) - I_{0,i}) W(\alpha_s) W(\alpha_i) d\alpha_s d\alpha_i. \end{aligned} \quad (19)$$

Once more, something like that can always be rewritten (see former section), for some suitable and positively defined $f'(x)$, as

$$P_i = \int_{\alpha_s} \int_{\alpha_i} f'(I_i(\alpha_s, \alpha_i)) W(\alpha_s) W(\alpha_i) d\alpha_s d\alpha_i. \quad (20)$$

Integrating on α_i we would obtain

$$P_i = \int_{\alpha_s} \hat{f}'_i(\alpha_s) W(\alpha_s) d\alpha_s, \quad (21)$$

from where we define a new function $\hat{f}'_i(\alpha_s)$. On the other hand, for joint detections we would have

$$\begin{aligned} P_{i,j} &\propto \int_{\alpha_s} \int_{\alpha_i} \int_{\alpha_j} (I_i(\alpha_s, \alpha_i) - I_{0,i}) \cdot (I_j(\alpha_s, \alpha_j) - I_{0,j}) \\ &\quad \times W(\alpha_s) W(\alpha_i) W(\alpha_j) d\alpha_s d\alpha_i d\alpha_j, \end{aligned} \quad (22)$$

which again can always be rewritten (again see former section), for some positively defined $\Gamma'(x, y)$, as

$$\begin{aligned} P_{i,j} &= \int_{\alpha_s} \int_{\alpha_i} \int_{\alpha_j} \Gamma'(I_i(\alpha_s, \alpha_i), I_j(\alpha_s, \alpha_j)) \\ &\quad \times W(\alpha_s) W(\alpha_i) W(\alpha_j) d\alpha_s d\alpha_i d\alpha_j, \end{aligned} \quad (23)$$

and integrating on α_i, α_j we would obtain

$$P_{i,j} = \int_{\alpha_s} \hat{\Gamma}'_{i,j}(\alpha_s) W(\alpha_s) d\alpha_s, \quad (24)$$

from where we can again define yet another new probability density function $\hat{\Gamma}'_{i,j}(\alpha_s)$.

It is interesting for the sake of clarity to compare the two situations: with (primed functions) and without polarizers (unprimed). It is easy to see that, because the detector only sees the intensity at its entrance channel, clearly (we drop the “s” subscript for simplicity),

$$f'(x) = f(x), \quad \forall x, \quad (25)$$

$$\Gamma'(x, y) = \Gamma(x, y), \quad \forall x, y. \quad (26)$$

while, naturally, in general $\hat{f}'_i(\alpha) \neq \hat{f}_i(\alpha)$, as well as $\hat{\Gamma}'_{i,j}(\alpha) \neq \hat{\Gamma}_{i,j}(\alpha)$.

IV. ON NON-FACTORABILITY

A. Mathematical analysis

Let us go back to the case with just the source and the detectors; we will soon see the following does nevertheless also apply when polarizers or other devices are added to the setup, just the same. According to our reasonings in App. 2, and using (15)–(16), we now realize that there is no way to avoid

$$\hat{\Gamma}_{i,j}(\alpha) = \hat{f}_i(\alpha) \cdot \hat{f}_j(\alpha), \quad (27)$$

unless we introduce some additional dependence of the kind $\hat{f}(\alpha) \rightarrow \hat{f}(\alpha, \mu)$, so that then

$$\hat{\Gamma}_{i,j}(\alpha, \mu) \neq \hat{f}_i(\alpha, \mu) \cdot \hat{f}_j(\alpha, \mu), \quad (28)$$

where we add a second “hat” to avoid an abuse of notation, and where μ stands for a new set of random variables. This $\hat{\hat{f}}(\alpha, \mu)$ should be interpreted as a detection probability conditioned to the new vector of random variables μ , i.e.,

$$\hat{\hat{f}}(\alpha, \mu) \equiv P(det|\alpha, \mu). \quad (29)$$

We will impose further demands on $\hat{\hat{f}}(\alpha, \mu)$, defining

$$\hat{\hat{f}}(\alpha, \mu) \equiv f(I(\alpha, \mu)), \quad (30)$$

something forced by strictly physical arguments: the choice $f(I(\alpha, \mu))$ must prevail over other possible ones - for instance $f(I(\alpha), \mu)$ - due to the need to respect the dependence of the probabilities of detection (conditioned to α or not) alone on the intensity that arrives to the detector, and nothing else. Now, with the density function

$\rho_\mu(\mu)$, we could write

$$P(det|\alpha) = \int_\mu P(det|\alpha, \mu) \rho_\mu(\mu) d\mu, \quad (31)$$

$$P_{i,j}(det|\alpha) = \int_\mu P_{i,j}(det|\alpha, \mu) \rho_\mu(\mu) d\mu, \quad (32)$$

allowing us to recover our former definitions (15)–(16):

$$\hat{f}(\alpha) \equiv P(det|\alpha), \quad (33)$$

$$\hat{\Gamma}_{i,j}(\alpha) \equiv P_{i,j}(det|\alpha). \quad (34)$$

For joint detections, the additional variable μ is particularly relevant because, we will always have that while

$$P_{i,j}(det|\alpha, \mu) = P_i(det|\alpha, \mu) \cdot P_j(det|\alpha, \mu), \quad (35)$$

in general

$$P_{i,j}(det|\alpha) \neq P_i(det|\alpha) \cdot P_j(det|\alpha), \quad (36)$$

or we could equivalently say that while necessarily

$$\hat{\Gamma}_{i,j}(\alpha, \mu) = \hat{f}_i(\alpha, \mu) \cdot \hat{f}_j(\alpha, \mu), \quad (37)$$

in general

$$\hat{\Gamma}_{i,j}(\alpha) \neq \hat{f}_i(\alpha) \cdot \hat{f}_j(\alpha), \quad (38)$$

where of course

$$\hat{\Gamma}_{i,j}(\alpha) = \int_\mu \hat{f}_i(\alpha, \mu) \cdot \hat{f}_j(\alpha, \mu) \rho_\mu(\mu) d\mu. \quad (39)$$

To conclude this section, we recover the case with intermediate devices: due to α_i, α_j being, as defined, independent from one another and also from α_s , our hypothetical “flag” μ cannot be associated with none of them. Therefore, in general, and in principle, not only

$$\hat{\Gamma}'_{i,j}(\alpha) \neq \hat{f}'_i(\alpha) \cdot \hat{f}'_j(\alpha), \quad (40)$$

but also $\hat{\Gamma}_{i,j}(\alpha) \neq \hat{f}_i(\alpha) \cdot \hat{f}_j(\alpha)$ either. We use “in principle” because this question is not yet analyzed in detail; we now see crystal clear, though, that this possible non-factorability on α ’s is nothing more than an internal feature of the model’s mathematical structure, bearing no relevance in regard to its double-sided compatibility (or absence of it) both with local-realism and the quantum predicted correlations. Indeed, the crucial element in that sense is the intervention of new random vacuum amplitudes introduced by the new devices in the setup, as we will see in Sec. V.

B. A physical interpretation

Recapitulating, we have that while $P_{i,j}(det|\alpha, \mu) = P_i(det|\alpha, \mu) \cdot P_j(det|\alpha, \mu)$, however $P_{i,j}(det|\alpha) \neq$

$P_i(det|\alpha) \cdot P_j(det|\alpha)$, where μ would be a new, additional HV (or vector of HV’s) and should, in the ultimate term, correspond to some physically relevant variable that we have so far overlooked, perhaps containing additional information about the state of the crystal and the laser. Indeed, if it was eventually necessary to introduce a new variable μ , physical intuition suggests that it could probably be associated to the phase of the laser complex amplitude [23]; with the laser in a coherent state, it would be justified to approximate $\rho_\mu(\mu) \approx \delta(\mu_0)$, where μ_0 would be the phase or the quasi-classical wave. Perhaps it is needless to add that in such a case expressions such as (39) or a similar one for Γ ’s would still be fully operative, as long as the dependence on μ_0 is “absorbed” in the functional dependence at the left side of the equation. Anyway, whether this μ has an obvious physical meaning or not should not bother us too much here, as it does not affect at all the correctness and/or the implications of our mathematical results.

V. COMPLEMENTARY QUESTIONS

A. Wigner-PDC’s local realism vs. quantum correlations

Though former mathematical developments are fully meaningful and self-contained on their own, yet it would be convenient to give some hints on how a local-realist (LR) model can account for typically quantum correlations, which are known to defy that very same local realism (LR). In the first place and as a general answer, what the Wigner-PDC picture proves is that LR is respected by a certain subset of all the possible quantum states, specifically the ones that can be generated from a non-linear mix of the QED-vacuum (which therefore acts as an “input” for the model) with a quasi-classical (a high-intensity coherent state), highly directional signal, the laser “pump” (which indeed enters in the model as a non-quantized, external potential [3]). Moreover, such a restriction is clearly not arbitrary at all, since it arises from a very simple quantum electrodynamical model of the process of generation of polarization-entangled pairs of photons from Parametric Down Conversion (PDC): see for instance eq. (4.2) in [3].

B. Detection rates and “efficiencies”

Aside from subscripts, we will now also drop “hats” and “primes” for simplicity; of course the fact that $\Gamma_{i,j}(\alpha)$ may not be in general α -factorisable,

$$\Gamma_{i,j}(\alpha) \neq f_i(\alpha) \cdot f_j(\alpha), \quad (41)$$

does not at all mean that it cannot well satisfy

$$\int \Gamma_{i,j}(\alpha) W(\alpha) d\alpha = \left[\int f_i(\alpha) W(\alpha) d\alpha \right] \cdot \left[\int f_j(\alpha') W(\alpha') d\alpha' \right], \quad (42)$$

i.e. (let us from now use superscripts “W” and “exp” to denote, respectively, theoretical and experimental detection rates):

$$P_{i,j}^{(W)}(det) = P_i^{(W)}(det) \cdot P_j^{(W)}(det), \quad (43)$$

which is indeed the sense in which the hypothesis of “error independence” is introduced, to our knowledge, in every work on LHV models [25]. This sort of conditions over “average” probabilities (average in the sense that they are integrated in the hidden variable, may that be α alone or also some other one) are the only ones that can be tested in the actual experiment; there, we can just rely on the number of counts registered on a certain time-window ΔT , and the corresponding estimates of the type

$$P_i^{(exp)}(det) \approx \frac{n. \text{ joint det. } (i, j) \text{ in } \Delta T}{n. \text{ marg. det. } (j) \text{ in } \Delta T}. \quad (44)$$

Now, if we wish to include some additional uncertainty element reflecting the technological limitations (a “detection efficiency” parameter), what we have to do is to redefine the overall detection probabilities as

$$P_i^{(exp)}(det) \equiv \hat{\eta}_i \cdot P_i^{(W)}(det), \quad (45)$$

$$P_{i,j}^{(exp)}(det) \equiv \hat{\eta}_i \hat{\eta}_j \cdot P_{i,j}^{(W)}(det), \quad (46)$$

as well as

$$P_i^{(exp)}(det|\alpha) \equiv \hat{\eta}_i \cdot P_i^{(W)}(det|\alpha), \quad (47)$$

$$P_{i,j}^{(exp)}(det|\alpha) \equiv \hat{\eta}_i \hat{\eta}_j \cdot P_{i,j}^{(W)}(det|\alpha), \quad (48)$$

where $0 \leq \hat{\eta}_i \leq 1$ would play the role of such an (alleged) detection efficiency, the “hats” remarking the fact that the customary definition of the analogous quantity in QInf involves not only our $\hat{\eta}$ ’s but also the non-technological contribution. From the point of view of the experimenter it is very difficult to isolate both components (Glauber’s theory [15] does not predict a unit detection probability even for high intensity signals); we should perhaps then confine ourselves to the term “observable detection rate” instead of using the clearly misleading one of “detector inefficiency”.

C. Consequences on Bell tests, their supplementary assumptions and critical efficiencies

That said and going to a lowest level of detail, states in such an (LR) subset of QED can still indeed exhibit

correlations of the class that is believed to collide with LR, yet the procedure through which they are extracted from the experimental set of data does not meet one of the basic assumptions required by every test of a Bell inequality: they do not keep statistical significance with respect to the physical set of “states” or hidden instructions [25]). To guarantee that statistical significance we must introduce some of the following two hypothesis:

(i) all coincidence detection probabilities are independent of the polarizers’ orientations ϕ_i, ϕ_j (this is what we call “fair-sampling” [26], for a test of an homogeneous inequality [27]), which implies

$$P_{ij}(det|\phi_i, \phi_j, \alpha) = P_{ij}(det|\alpha_s), \quad (49)$$

where of course (see Sec. III) $\alpha \equiv \alpha_s \oplus \alpha_i \oplus \alpha_j$, and where we recall that ϕ_i, ϕ_j would determine which vacuum modes amongst the sets α_i, α_j would intervene in the detection process,

(ii) the interposition of an element between the source and the detector cannot in any case enhance the probability of detection (the “no-enhancement” hypothesis [28], needed to test the Clauser-Horne inequality [19], and presumably every other inhomogeneous one [29]),

$$P_i(det|\phi_i, \alpha) \leq P_i(det|\infty, \alpha_s). \quad (50)$$

with ∞ denoting the absence of polarizer and with $\alpha \equiv \alpha_s \oplus \alpha_i$ this time.

Following our developments in Sec. III one can easily see that (i) is not in general true, and according to [13] neither is (ii). I.e., whenever states of light are prepared so as to produce the sort of quantum correlations that are known to defy LR, these last come supplemented with the necessary features that prevent it from happening... how could it not?

Yet, a mere breach of (i) or (ii) is not enough to assert the existence of a Local Hidden Variables (LHV) model, which is an equivalent way of saying that the results of the experiment respect LR: it is more than well known that this can only happen for certain values of the observed detection rates [25, 30]. However, from the point of view of (47)–(48), and given the fact that, as proven in Secs II and III, the Wigner-PDC is in all circumstances in accordance with expressions (54)–(55), and hence to all possible Bell inequalities whether our $\hat{\eta}$ ’s are equal or less than unity, the so-called “critical efficiencies” would merely stand, at least as far as PDC-generated photons are concerned, for bounds on the detection rates that we can experimentally observe (these last in turn constrained by the only subset of quantum states that we can physically prepare).

D. An approach to realistic detectors and average ZPF subtraction

In this section we make a slight approach on some physical considerations with the aim of showing that there is plenty of room for a suitable physical interpretation of the model, even when that is not strictly necessary for the coherence of our results, at least from the purely mathematical point of view. Indeed, we have already descended to the physical level when we established, in former sections, the dependence of our f 's and Γ 's solely on the intensity arriving at the detector.

Expectable behavior for a physical device would typically include a “dead-zone”, an approximately linear range and a “saturation” at high intensities (this is indeed the kind of behavior suggested for instance in [16]); amongst other restrictions this would imply, for instance, $f(I(\alpha)) = 0$, when $I(\alpha) \leq \bar{I}_0$ (this last a threshold that may even surpass the expectation value of the ZPF intensity), as well as $\Gamma(I_i(\alpha), I_j(\alpha)) = 0$ either for $I_i(\alpha)$ or $I_j(\alpha)$ below \bar{I}_0 . Neither these restrictions nor other similar ones would in principle invalidate our proofs in Appendix 1, which seem to provide room enough to simulate a wide range of possible behaviors; however, we must remark that none of our f 's and Γ 's can ultimately be considered as fully physical models, due to the fact that they represent point-like detectors (the implications of such an over-simplification may become clearer in the light of our following point here).

In close relation to the former, we also propose here a simple physical interpretation of the term $-I_0$ appearing in the expressions for the detection probabilities. From the mathematical point of view such subtraction arises from a mere manipulation of Glauber's original expression [15]. From the physical one, a realistic interpretation would be more than desirable, as that subtraction of ZPF intensity is for instance crucial to explain the absence of an observable contribution of the vacuum field on the detectors' rates [31]; of course we mean “explain in physical terms”; from the mathematical point of view our model here already predicts a vanishing detection probability for the ZPF alone. Our suggestion is that the $-I_0$ term must be (at least) related to the average flux of energy going through the surface of the detector in the opposite direction to the signal (therefore *leaving* the detector). That interpretation fits the picture of a detector as a physical system producing a signal that depends (with more or less proportionality on some range) on the total energy (intensity times surface times time) that it accumulates. For the moment we just say “at least related”, bearing in mind that to establish such association we would first have to refine the point-like model of a detector which stems directly from the original Glauber's expression [15].

VI. CONCLUSIONS

We have shown that the Wigner description of PDC-generated (Parametric Down Conversion) photon-entanglement, so enthusiastically developed in the late nineties [3–9] but then ignored in recent years, can be formulated as entirely local-realistic (LR). A formalism that is one-to-one with a quantum (field-theoretical) model of the experimental setup can be cast, thanks to an additional manipulation (also one-to-one), into a form that respects all axiomatic laws of probability [32], and therefore LR, as defined for instance in Sec. 2a. The original quantum electrodynamical model takes as an input the vacuum state, which accepts a well defined probabilistic description through the so-called Wigner transform: this is the fact that the analogy with a local-realistic theory is conditioned to. What we call Wigner-PDC accounts, then, for a certain subset within the space of all possible QED-states, determined by a particular set of initial conditions and a certain Hamiltonian governing the time-evolution [40], restrictions that seem to guarantee (according to us here) the compatibility with LR.

Aiming for the maximum generality, as well as to avoid some possible (still under examination) difficulties with factorisable expressions, we have renounced to what we call α -factorability of the joint detection expressions. Such a choice is not only perfectly legitimate [20], but may also be supported by a well feasible interpretation (see Appendix IV); nevertheless, further implications of that non-factorability on α will also be left to be examined elsewhere. Again, whatever they finally turn out to be, they are also irrelevant for our main result in this paper: the Wigner-PDC formalism can be cast into a form that respects all axiomatic laws of probability for space-like separated events.

Neither does the explicit distinction between the cases where polarizers (or other devices placed between the source and the detectors) are or not included in the setup introduce any conceptual difference from the point of view of our main result; however, the question opens room to remark some of the main differences of the Wigner-PDC model (actually, also its Hilbert-space analogue) with the customary description used in the field of Quantum Information (QInf): here, each new device introduces noise, new vacuum amplitudes that fill each of its empty polarization channels at each of its exists, in contrast to the usual QInf “black box”, able to extract polarization information from a photon without any indeterministic component.

As a matter of fact, those additional random components hold the key to explain the variability of the detection probability (see note [13]) that is necessary, from the point of view of Bell inequalities, to reconcile quantum predictions and LR (see Sec. V A). In particular, the immediacy with which the phenomenon of “detection probability enhancement” arises in the Wigner-PDC framework would suggest that this may be after all the right track to understand why after several decades the min-

imum detection rates (or, in QInf terminology, critical detection efficiencies) that would lead to obtain conclusive evidence of non-locality are yet to be reached [30].

In Sec. VD we have done a first, general approach on the question of whether our reinterpretation of the detection probabilities is consistent with the actual physical behavior of detectors. A closer look to this question is nevertheless left out of the scope of the paper; former proposals [16] in regard to this issue aimed perhaps too straightforwardly to the physical level, while they did not even guarantee consistency with the framework we have settled here. In close relation with the former, we have also suggested a possible simple physical interpretation of I_0 -subtraction taking place in the expressions for the detection probabilities: work in any of this two directions would anyway require a departure from the point-like model of a detector on which we have focused here. Summarizing, we have shown that a whole family of detection models

$$\mathcal{M}_{det} \equiv \{f(I(\alpha, \mu)), \Gamma(I_i(\alpha, \mu), I_j(\alpha, \mu)), \quad (51)$$

can be found, consistent both with the quantum mechanical expressions from the Wigner-PDC model and LR. A close examination of the constraints coming from the physical behavior of the detectors and other experimentally testable features is left as a necessary step for the future, with the aim of establishing a subset physically feasible ones; nevertheless, we have the guarantee that all of them produce suitable predictions, as so does their quantum electrodynamical counterpart.

Yet, even at the purely theoretical level some other features remain open too: as a fundamental one, to what extent the model requires what we have called non-factorability on α 's. Anyway and as a final conclusion, though our work here does not imply that we have already achieved an entirely proficient description of PDC-based photon entanglement, at least for us is already evident that, up to the point we have come up to, that description is indeed completely local-realistic.

It is more than proven, yes, that QM is sometimes incompatible with local-realism, but... is really every quantum state physically realizable? Does QM describe nature or is it just a subset of it (the one that is compatible with local-realism) that really does? After all, QM is just a theory, a theory that provides a formalism upon which to build models, models than can (and should) be refined based on experimental evidence; quantum entanglement seems to manifest in many of its “reasonable” features but, at least as PDC is concerned and so far, whenever local-realism would seem to be challenged new phenomena appear preventing it. Such are “unfair sampling”, “enhancement” (as a particular case of variable detection probability) and over all detection rates low enough to open room for the former two. To explore, and exhaust if that is the case, alternative routes such as the one here is not only sensible but also necessary.

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Appendix

1. Auxiliary proofs

Lemma 1: There always exists $f(x)$ satisfying (9), with $0 \leq f(x) \leq 1$, for $x = I(\alpha, \alpha^*)$ and all possible pairs α, α^* .

Proof: We have a linear system with one restriction and infinite independent variables: N^2 ($N \rightarrow \infty$) real numbers $f(I(\alpha, \alpha^*))$, one per each pair α, α^* . Coefficients are given by the $W(\alpha, \alpha^*)$'s, and the independent term is the left-side term of (9), $P_s \geq 0$. We consider the real, vectorial space \mathcal{V} where a particular solution of the problem is represented associating one $f(I(\alpha, \alpha^*))$ per coordinate: $\dim(\mathcal{V}) = N^2$ (again with $N \rightarrow \infty$). If they do exist, compatible solutions $f(I(\alpha, \alpha^*))$ for the (unique) restriction will conform a linear manifold $\mathcal{M} \subset \mathcal{V}$. Now we have to see:

(i) Due to $W(\alpha, \alpha^*) > 0$ for all pairs α, α^* [33], and $P_s \geq 0$ too, \mathcal{M} cannot be parallel with any of the coordinate hyper-planes in \mathcal{V} ; i.e., \mathcal{M} intersects all of them, defined each (each one for a pair α, α^*) by the equation $f(I(\alpha, \alpha^*)) = 0$.

(ii) Moreover, for $P_s > 0$, \mathcal{M} has a non-trivial intersection (more than one point, the origin) with all coordinate planes inside the first hyper-quadrant \mathcal{V}_{1q} (the subregion $\mathcal{V}_{1q} \subset \mathcal{V}$ given by restricting \mathcal{V} to $f(I(\alpha, \alpha^*)) \geq 0$). This can be seen, for instance, determining the point of crossing with the axes: doing all f 's zero except one (always possible because all the W 's are strictly above zero), we can see the crossings always take place at the positive half of the corresponding coordinate axis, therefore at the boundary of \mathcal{V}_{1q} . For $P_s = 0$, the solution for the system is trivial.

With (i) and (ii), it is clear that under the restriction (under the set of inequalities) $f(I(\alpha, \alpha^*)) \leq 1 \forall \alpha, \alpha^*$, the set of admissible solutions is still not empty. This is guaranteed with

$$\int_{\alpha, \alpha^*} W(\alpha, \alpha^*) d\alpha d\alpha^* = 1. \quad (52)$$

and $P_s \leq 1$. Here we give an inductive reasoning; think in 3 dimensions: what we have is $ax + by + cz = d$, with $a, b, c > 0$ and $d \geq 0$; the coordinates of the point of intersection with $x = y = z$ are given (obviously all) by the quantity $d/(a + b + c)$. The extension to infinite dimensions, topological abnormalities all absent, is direct,

with $a, b, c \equiv W's$, $d \equiv P_s$ and $a + b + c \equiv 1$, and finally $P_s \leq 1$, guaranteeing the existence of valid solutions. ■

Lemma 2: There always exists $\Gamma(x, y)$ satisfying (12), with $0 \leq \Gamma(x, y) \leq 1$, for $x = I_i(\alpha, \alpha^)$, $y = I_j(\alpha, \alpha^*)$ and all possible pairs α, α^* .*

Proof: formally identical with Lemma 1. ■

Lemmas 1 and 2 are directly applicable, aside from Sec. II, to Sec. III.

2. Basic concepts revisited

We first briefly revisit the concepts of locality, determinism and factorability; a good understanding on these concepts is crucial for the main results of the paper, what makes this review not only convenient but almost unavoidable, especially given the presence of some confusion in the literature. In any case, it shall be clearly understood that *Clauser and Horne's factorability [19] is not a requisite for local-realism*.

a. Locality and realism

A theory predicting the results of two measurements A and B that take place under causal disconnection (relativistic space-like separation) can be defined as local if and only if we can write

$$A = A(\lambda, \phi_A), \quad B = B(\lambda, \phi_B), \quad (53)$$

where λ is a (set of) hidden (or explicit) variables defined inside the intersection of both light cones, and ϕ_A, ϕ_B are another two other sets of variables (amongst them the configurable parameters of the measuring devices) defined locally at A and B , respectively, and causally disconnected from each other, i.e.,

$$P(A = a | \lambda, \phi_A, \phi_B) = P(A = a | \lambda, \phi_A), \quad (54)$$

$$P(B = b | \lambda, \phi_A, \phi_B) = P(B = b | \lambda, \phi_B). \quad (55)$$

These last two expressions are usually taken as a definition of *local causality* [34].

Now, realism simply stands for λ (and ϕ_A, ϕ_B as well) having a well defined probability distribution. A set of physical observables corresponding to a particular quantum state can be sometimes described by a well defined joint probability density (such is the case of field amplitudes in any point of space for the vacuum state in QED); for other quantum states that is not possible though.

b. Determinism

A measurement M upon a certain physical system, with k possible outcomes m_k , is *deterministic* on a hidden

variable (HV) λ (summarizing the state of that system), if (and only if)

$$P(M = m_k | \lambda) \in \{0, 1\}, \quad \forall k, \lambda, \quad (56)$$

which allows us to write

$$M \equiv M(\lambda), \quad (57)$$

and *indeterministic* iff, for some λ , some k' ,

$$P(M = m_{k'} | \lambda) \neq \{0, 1\}, \quad (58)$$

i.e., at least for some (at least two) of the results for at least one (physically meaningful) value of λ .

Now, indeterminism can be turned into determinism, i.e., (58) can be turned into (56), by defining a new hidden variable μ , so that now, with $\gamma \equiv \lambda \oplus \mu$:

$$P(M = m_k | \gamma) \in \{0, 1\}, \quad \forall k, \gamma, \quad (59)$$

which means we can write,

$$M \equiv M(\lambda, \mu), \quad (60)$$

a proof that such a new hidden variable μ can always be found (or built) given in [35].

So far, then, our *determinism* and *indeterminism* are conceptually equivalent, though of course they may correspond to different physical situations, depending for instance on whether γ is experimentally accessible or not.

c. Factorability

Let now $\mathcal{M} = \{M_i\}$ be a set of possible measurements, each with a set $\{m_{i,k}\}$ of possible outcomes, not necessarily isolated from each other by a space-like interval. We will introduce the following distinction: we will say \mathcal{M} is

(a) *λ -factorisable*, iff we can find a set $\{\xi_i\}$ of random variables, *independent from each other and from λ too*, such that

$$\mu = \bigoplus_i \xi_i, \quad (61)$$

and (56) holds again for each M_i on $\gamma_i \equiv \lambda \oplus \xi_i$:

$$P(M_i = m_{i,k} | \gamma_i) \in \{0, 1\}, \quad \forall i, \forall k, \gamma_i, \quad (62)$$

this last expression meaning of course that we can write, again for any of the M_i 's,

$$M_i \equiv M_i(\lambda, \xi_i). \quad (63)$$

(b) *non λ -factorisable*, iff (62) is not possible for any set of statistically independent ξ_i 's.

We will restrict, for simplicity, our reasonings to just two possible measurements $A, B \in \mathcal{M}$, with two possible

outcomes, $A, B \in \{+1, -1\}$, all without loss of generality. We have seen that, as the more general formulation, we can always write something like $A = A(\lambda, \xi_A)$, $B = B(\lambda, \xi_B)$.

Lemma 1–

(i) *If A and B are deterministic on λ , i.e., (56) holds for A and B , then they are also λ -factorisable, i.e.,*

$$P(A = a, B = b|\lambda) = P(A = a|\lambda) \cdot P(B = b|\lambda), \quad (64)$$

for any $a, b \in \{+1, -1\}$. Eq. (64) is nothing but the so-called Clauser-Horne factorability condition [19].

(ii) *If A and B are indeterministic on λ , i.e., if (56) does not hold for λ , then: for some μ (always possible to find [35]) such that now (59) holds for $\gamma \equiv \lambda \oplus \mu$, A, B are γ -factorisable,*

$$P(A = a, B = b|\gamma) = P(A = a|\gamma) \cdot P(B = b|\gamma), \quad (65)$$

i.e., (64) holds for γ , this time not necessarily for λ .

(iii) Let (62) hold for A, B , on λ, ξ_A, ξ_B : if λ, ξ_A, ξ_B are statistically independent, (hence, A and B are what we have called λ -factorisable), then (64) holds for λ , not necessarily on the contrary.

Proof–

(i) When (56) holds, for any λ and any $a, b \in \{+1, -1\}$, $P(A = a|\lambda), P(B = b|\lambda) \in \{0, 1\}$, from where we can, trivially, get to (64). ■

(ii) It is also trivial that, if (59) holds, (64) can be recovered for γ . That the same is not necessary for λ can be seen with the following counterexample: suppose, for instance, that for $\lambda = \lambda_0$, either $A = B = 1$ or $A = B = -1$ with equal probability. It is easy to see that

$$P(A = B = 1|\lambda_0) \neq P(A = 1|\lambda_0) \cdot P(B = 1|\lambda_0), \quad (66)$$

numerically: $\frac{1}{2} \neq \frac{1}{4}$. ■

(iii) We have, from independence of λ, ξ_A, ξ_B , and working with probability densities ρ 's: $\rho_\lambda(\lambda, \xi_A, \xi_B) = \rho_\lambda(\lambda) \cdot \rho_A(\xi_A) \cdot \rho_B(\xi_B)$, which we can use to write

$$\begin{aligned} P(A = a, B = b|\lambda) &= \int P(A = a, B = b|\lambda, \xi_A, \xi_B) \\ &\quad \times \rho_A(\xi_A) \cdot \rho_B(\xi_B) d\xi_A d\xi_B. \end{aligned} \quad (67)$$

and now with the fact that we can recover (59) for A (B) on $\gamma_A = \lambda \oplus \xi_A$ ($\gamma_B = \lambda \oplus \xi_B$),

$$\begin{aligned} P(A = a, B = b|\lambda) &= \int P(A = a|\lambda, \xi_A) \cdot P(B = b|\lambda, \xi_B) \\ &\quad \times \rho_A(\xi_A) \cdot \rho_B(\xi_B) d\xi_A d\xi_B \\ &= \int P(A = a|\lambda, \xi_A) \cdot \rho_A(\xi_A) d\xi_A \\ &\quad \times \int P(B = b|\lambda, \xi_B) \cdot \rho_B(\xi_B) d\xi_B \\ &= P(A = a|\lambda) \cdot P(B = b|\lambda). \end{aligned} \quad (68)$$

On the other hand, let λ, ξ_A, ξ_B be not statistically independent: we can set for instance, as a particular case, $\xi_i \equiv \mu$, $\forall i$, therefore reducing our case to that of (59). Once this is done, our previous counterexample in (ii) is also valid to show that factorability is not necessary for λ here. ■

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 - [10] A. Casado, S. Guerra, J. Plácido. *J. Phys. B: At. Mol. Opt. Phys.* **41**, 045501 (2008).
 - [11] A. Casado, S. Guerra, J. Plácido. *Advances in Mathematical Physics* 501521 (2010).
 - [12] The Wigner transform applies a quantum state ρ in a real, multivariable function:

$$W[\rho] : \rho \rightarrow W(\{x, p\}_{\mathcal{H}}) \in \mathcal{R}, \quad (69)$$

where the set $\{x, p\}_{\mathcal{H}}$ of variables upon which $W(\cdot)$ takes values depends on the structure of \mathcal{H} , the (quantum mechanical) Hilbert space of the system. Under certain restrictions (for a subset of all possible quantum states), $W[\rho]$ can be interpreted as a (joint) probability density function. For instance see:

Y.S. Kim, M.E. Noz. “Phase space picture of Quantum Mechanics (Group theoretical approach)”, *Lecture Notes in Physics Series* - Vol **40**, ed. World Scientific.

- [13] To explain the enhancement of the detection probability as a result of the interposition of a polarizer between the source and the detector, we have to take into account that polarization components reflected by the polarizer must be replaced, at the entrance of the detector, by new components given by vacuum amplitudes at that point. As a difference with macroscopical light where enhancement is an obvious phenomenon that does not require ZPF considerations at any step, when we have only one photon it becomes impossible unless we include the vacuum fluctuations in the picture.

For instance, from E. Santos, “Photons Are Fluctuations of a Random (Zeropoint) Radiation Filling the Whole Space”, in *The Nature of Light: What Is a Photon?*, p.163, edited by Taylor & Francis Group, LLC (2008): “one of the additional hypotheses used, introduced by Clauser and Horne with the name of “no-enhancement”, is naturally violated in SO because a light beam crossing a polarizer may increase its intensity, due to the insertion of ZPF in the fourth channel (...), which is the possibility excluded by the no-enhancement assumption”.

- [14] A. Casado, private communication.
 [15] The expressions for the detection probabilities in the Wigner-PDC picture can be obtained by mere manipulation of Glauber’s original expressions for marginal and joint detections, respectively:

$$P_i \propto \langle \psi | \hat{E}_i^{(-)} \hat{E}_i^{(+)} | \psi \rangle, \quad (70)$$

$$P_{i,j} \propto \langle \psi | \hat{E}_i^{(-)} \hat{E}_j^{(-)} \hat{E}_j^{(+)} \hat{E}_i^{(+)} | \psi \rangle. \quad (71)$$

with $|\psi\rangle$ representing the state of the fields in all space, the field operator $\hat{E}_i^{(-)}$ ($\hat{E}_i^{(+)}$) defined at the position of the i -th detector \mathbf{r}_i and containing only creation (annihilation) operators, therefore giving rise to expressions in the so-called *normal order* of quantum operators. See [3, 6] for details.

- [16] A. Casado, T.W. Marshall, R. Risco-Delgado, E. Santos. “A local hidden variables model for experiments involving photon pairs produced in parametric Down Conversion”, arXiv:quant-ph/0202097v1 (2002).
 Roughly speaking, their proposal can be understood as a substitution of our function $f(I(\alpha)) \equiv f(I(\alpha) - I_0)$ by a new one

$$f_{trial}(I(\alpha) - I_0(\alpha)) \equiv \hat{f}_{trial}(\alpha) \equiv (1 - e^{-\zeta [\bar{I}(\alpha) - \bar{I}_0(\alpha)]}) \cdot \Theta [\bar{I}(\alpha) - I_m], \quad (72)$$

where ζ, I_m are parameters, Θ is the Heaviside function, $I_m > I_0$ represents a “threshold” intensity and $\bar{I}(\alpha), \bar{I}_0(\alpha)$ correspond to our intensities $I(\alpha), I_0(\alpha)$, except for the fact that they are calculated performing a previous spatial and temporal integration (on the respective angular and temporal windows of observation) of the field complex amplitude.

The differences between the former proposal and our approach are two. First, the inclusion of spatial/temporal integration; regardless of its relevance in relation to the physical behavior of detectors [31] (we insist that that is not our focus here), the (quantum electrodynamical) model from where we depart (for instance see eq. 4.2 in [3]) involves indeed a “point-like” model of a detector: it is therefore at that stage where proper refinements should be done, and we justified in leaving the question aside for further works. As a second one, f_{trial} deviates from the quantum predictions both at low and high intensities [22], consistently with the fact that neither f_{trial} nor the models in [17] belong to our class of acceptable functions f (guaranteeing a one-to-one correspondence with the initial quantum electrodynamical model).

Indeed, we are left to wonder how our much less ambitious but certainly necessary step was not properly attacked before; it is our view that a project which involves unsolved problems both at the purely mathematical and physical levels should be approached (at least) in two steps. Here we have taken the first of them (regarding the purely mathematical issues); the second would stand for applying all the necessary constraints to reproduce the actual physical behavior of detectors.

- [17] T.W. Marshall, E. Santos, “A classical model for a photo-detector in the presence of electromagnetic vacuum fluctuations”, arXiv:0707.2137.
 [18] For all t, \mathbf{r} , one particular mode of the ZPF, with \mathbf{k} the wave-vector and γ the two possible orthogonal polarization states, is always expressible as

$$\mathbf{E}_0(\alpha, \alpha^*) = \sum_{\mathbf{k}, \gamma} \left[\alpha_{\mathbf{k}, \gamma} e^{-i(\omega t - \mathbf{k} \cdot \mathbf{r})} + \alpha_{\mathbf{k}, \gamma}^* e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \right], \quad (73)$$

where the (stochastic) amplitude $\alpha_{\mathbf{k}, \gamma}$ determines, for all space, a mode (\mathbf{k}, γ) . Of course, we can always define

$$\alpha_{\mathbf{k}, \gamma}(\mathbf{r}) = \alpha_{\mathbf{k}, \gamma}(\mathbf{r} = 0) e^{i\mathbf{k} \cdot \mathbf{r}}, \quad (74)$$

so that now, for a certain mode ω and at a particular point of space:

$$\mathbf{E}_0(\mathbf{r}) = \sum_{\mathbf{k}, \gamma} \left[\alpha_{\mathbf{k}, \gamma}(\mathbf{r}) e^{-i\omega t} + \alpha_{\mathbf{k}, \gamma}^*(\mathbf{r}) e^{i\omega t} \right], \quad (75)$$

where it is obvious that the amplitudes $\alpha_{\mathbf{k}, \gamma}(\mathbf{r})$ follow exactly the same probability distribution as $\alpha_{\mathbf{k}, \gamma}$, i.e., their distribution is governed by $W(\alpha, \alpha^*)$. This all said and for the sake of simplicity, we shall abuse notation (as is done customarily in [3–9]) with

$$\alpha, \alpha^* \equiv \{\alpha_{\mathbf{k}, \gamma}, \alpha_{\mathbf{k}, \gamma}^*\} \quad (76)$$

i.e., α, α^* denotes a set of mode amplitudes (therefore in principle defined for all space) for a given ω , (one or a set of) relevant wave-vectors \mathbf{k} and (the relevant) polarizations components γ . Finally, of course the overall vacuum field is obtained as a sum of all modes.

- [19] J.F. Clauser, M.A. Horne. Phys. Rev. D **10**, 526 (1974).
 [20] The so-called *factorability* condition [19], which for whatever two space-like separated observables A, B would read, in our notation,

$$P(A = a, B = b | \lambda) = P(A = a | \lambda) \cdot P(B = b | \lambda), \quad (77)$$

is not, as sometimes assumed, a necessary one ($\forall a, b, \lambda$), unless the outcomes of the measurements (either numerical ones or simply whether there is going to be a detection or not) are completely deterministic on λ .

For instance, suppose that for $\lambda = \lambda_0$, either $A = B = 1$ or $A = B = 0$ with equal probability; it is easy to see that $P(A = B = 1|\lambda_0) \neq P(A = 1|\lambda_0) \cdot P(B = 1|\lambda_0)$ ($\frac{1}{2} \neq \frac{1}{4}$).

We have addressed thoroughly this question in App.2; anyway and as shown there, the assumption of (77) in the case of [19] is nevertheless not criticizable, since one can always find (or define) a new hidden variable that guarantees it.

- [21] In the present framework, each wave-vector \mathbf{k} defines a plane wave propagating through the entire coordinate space, which obliges us to take some precautions. Let $\{\mathbf{k}_s\}$ and $\{\mathbf{k}_i\}$ be the sets of relevant wavevectors for the sets of relevant amplitudes $\{\alpha_s\}$ and $\{\alpha_i\}$, respectively, then we will suppose that $\{\mathbf{k}_s\} \cap \{\mathbf{k}_i\} = \emptyset$; in the (really very unlikely) case that this did not happen, we can always redefine α_i so the independence with $\{\alpha_s\}$ still holds.
- [22] For low intensities, the departure from the quantum mechanical model stands for a certain “dead-zone” and a much higher dark count rate.
At first order in perturbation theory, the expectation value of the number of photons is zero and therefore such rate vanishes: experimentally observed dark counts are usually associated to thermal effects or electronic noise. Besides, a saturation effect is introduced, which on the other hand is a feature of all real detectors, but in this case it appears at much lower intensities (for energies lower than one photon). Consistency with experimental data demands compliance with certain relations provided in that same paper [16].
- [23] Indeed, the laser is described by a coherent state, which for a high intensity signal means that it can be regarded as quasi-classical wave with well defined amplitude and phase (and introduced as a non-quantized external potential, see [3] for instance).
The phase of this complex amplitude is for instance relevant in expressions (4.10) from [3], and has the potential to interfere constructive or destructively with the α ’s, increasing or decreasing the overall intensity of the signal, therefore modifying the detection probabilities. From this point of view, such phase cannot be at all excluded from the vector of relevant hidden variables and therefore non-factorability (of detection probabilities) in α is the most natural feature to expect (though not strictly necessary), see our detailed analysis in Sec. IV.
- [24] R. Risco-Delgado, private communication.
- [25] For reference on how, and under which hypothesis LHV models are built:
 - (i) A. Cabello, D. Rodríguez, I. Villanueva. Phys. Rev. Lett. **101**, 120402 (2008),
 - (ii) A. Cabello, J. -Å. Larsson, D. Rodríguez. Phys. Rev. A **79**, 062109 (2009).
- [26] The *fair sampling* assumption is already formulated in [37]: “given a pair of photons emerges from the polarizers, the probability of their joint detection is independent of the polarizer orientations”.
- [27] We believe the distinction between *homogeneous* and *inhomogeneous* inequalities was first introduced by M. Horne himself (*ref*), then later updated by E. Santos

(*ref*). We will supersede previous definitions with our own here: we will call *inhomogeneous* inequalities all the ones involving coincidence rates of different order (for instance, marginal and joint ones), *homogeneous* on the contrary.

Of course, if for instance a certain subset of the observables involved in the inequality does not have any uncertainty associated to its detection, the inclusion of inhomogeneous terms may not lead to the requirement of supplementary assumptions; this is not the case of experiments with photons, anyway.

- [28] The *no-enhancement* assumption [19]: “for every emission λ , the probability of a count with a polarizer in place is less or equal to the probability with the polarizer removed”, where λ is the (hypothetical) hidden variable expressing the state (at least the initial one) of the pair of particles. We remark: “for every emission λ ”.
- [29] An inhomogeneous Bell inequality [27] requires the estimation of coincidence rates of different order (for instance single and double, or double and triple); due to the fact that there is no way to identify if the whole set of particles have been simultaneously emitted and then gone undetected, or they were simply never emitted at all, the test would always require supplementary assumptions; of course from a wave-like perspective, the issue is even more evident. In any case, marginal detection rates would not be experimentally accessible unless we assume “independent errors” at the detector for every and each of the “states” in the model (i.e., detection probabilities cannot depend on the hidden variables: α ’s, other such as the phase of the pump. etc).
- [30] Let us consider a Bell experiment and let η what people in QInf define as “detection efficiency” [25] (from our point of view this is clearly a misleading term, we should call it simply “detection rate”, whether is reduced or not by technological imperfections, see Sec. V A). A possible Local Hidden Variables (LHV) model for the experiment is then composed by a set of states $\{s\}$ with probabilistic weights (or frequencies) ρ_s : all restrictions the model must satisfy (either regarding the behavior of detectors or the quantum mechanical predictions) are linear in $\{\rho_s\}$ and the trivial solution $\eta = 0$ (all $\rho_s = 0$ but for the one that predetermines no detections at all) is always admissible, implying that there is always some $\eta_{\text{crit}} \geq 0$ such that for all $\eta \leq \eta_{\text{crit}}$ the desired LHV can be built. The reasoning applies whether if the LHV simulating just simulates a violation of a particular Bell inequality or also every other quantum prediction for a given state and set of observables (though the addition of more restrictions may lower η_{crit} , as expectable).
- [31] Though from the mathematical point of view our model here already predicts a vanishing detection probability for the ZPF alone, a physical interpretation of the absence of an observable detection rate as a result of the vacuum fluctuations, or at least that of a significant one, is still an open question. Following Santos’s work, the absence of the ZPF from the observational spectrum could be justified, for realistic detectors, on the combination of the following properties:
 - (i) the already commented subtraction of the average ZPF intensity as a result of Glauber’s expression in the normal order [15],
 - (ii) spatial and temporal integration, involving the autocorrelation properties of the ZPF,

(iii) a “low band pass” frequency response.

See, for instance: Emilio Santos, “How photon detectors remove vacuum fluctuations”, <http://arxiv.org/abs/quant-ph/0207073v2> (2008).

We remark again that this issue is in any case irrelevant for our main results here: it concerns models built outside of the quantum framework (which is not our case). Nevertheless, we conjecture that our proposed interpretation of (i) in Sec. VI may be of use to make progress on the question.

- [32] Let Λ be a space of events, all possible probability assignments within the formalism, $F(\cdot) : \lambda \in \Lambda \rightarrow F(\lambda)$, satisfy $0 \leq F(\lambda) \leq 1 \forall \lambda$ and, for any partition $\{\Lambda_i\}$ of Λ (assuming of course additivity on disjoint subsets of events), $\sum_i F(\Lambda_i) = 1$.

All Bell inequalities (at least those written in terms of probabilities) can be obtained as a derivation from these laws, plus some very simple supplementary assumptions: statistical independence of variables defined at distant locations. Of course, Bell inequalities written in terms of correlations can be also written in terms of probabilities. Many do not even need such supplementary assumptions (space-like separation ones or of another kind): such is the case of both the CH [19] and the CHSH [37] inequalities.

To illustrate this see for instance [38]’s derivation of the CH inequality; the CHSH inequality can on the other hand also be interpreted as a mere algebraic inequality on whatever four quantities taking values ± 1 .

- [33] $W(\alpha, \alpha^*)$ is a Gaussian [3]; hence $W(\alpha, \alpha^*) > 0 \forall \alpha, \alpha^*$.
- [34] For a recent work reviewing these concepts see for instance: T. Norsen, “J.S. Bell’s Concept of Local Causality”, <http://arxiv.org/abs/0707.0401>.
- [35] A possible (not unique) procedure to build μ is this: for each M_i , we define a new random variable σ_i and assign values for each pair $(\lambda, m_{i,k})$: $\sigma_i \equiv \sigma_i(\lambda, m_{i,k})$, and now simply do $\mu \equiv \bigoplus_i \sigma_i$. As built, σ_i ’s are not necessarily independent from one another, nor are they necessarily independent from λ .
- [36] A. Casado, private communication.
- [37] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt. Phys. Rev. Lett. **23**, 880 (1969).
- [38] J.-Å. Larsson and J. Semitecolos, Phys. Rev. A **63**, 022117 (2001).
- [39] R. Risco-Delgado. PHD Thesis. Universidad de Sevilla (1997).
- [40] The PDC model is by construction restricted to a certain subset of QED-states, obtained directly from a mix of the vacuum state (hence one with positive Wigner function) with a quasi-classical signal (the laser), and a time-evolution governed by a quadratic Hamiltonian (hence one that preserves the positivity of the Wigner function); for this last point, see for instance A.Casado, PHD Thesis: “Estudio de los experimentos de conversión paramétrica a la baja con el formalismo de la función de Wigner” (Universidad de Sevilla, 1998).